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## Quaternions and M(atrix) theory in spaces with boundaries

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### Abstract of this paper hep-th/9612198

A proposal for the matrix model formulation of the M-theory on a space with boundary is given. A general machinery for modding out a symmetry in M(atrix) theory is used for a  $Z_2$  symmetry changing the sign of the  $X_1$  coordinate. The construction causes the elements of matrices to be equivalent to real  $2 \times 2$  real blocks or quaternions and the symmetry  $U(2N)$  of the original model is reduced to  $O(2N)$  or  $USp(2N) = U(N, H)$ . We also show that membranes end on the boundary of the spacetime correctly in this construction.

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## Contents

1. Introduction
2. Modding out a symmetry in superstring theories
3. Modding out a symmetry in M(atrix) theory
4. The symmetry reversing spacetime and the membrane
  - Where do the membranes end?
  - Möbius and Klein bottle membranes
5. Gauging this symmetry and the appearance of quaternions
  - A short description of the  $U(1, H)$  system
  - The orthogonal choice
6. Conclusions

## 1 Introduction

Recently the first candidate for a non-perturbative formulation of the theory underlying all the superstring theories, branes, dualities, D-branes and so on has been given [1] in the form of infinite-N limit of a maximally supersymmetric matrix quantum mechanics. This formulation offers an infinite-momentum-frame formulation of a theory of  $N$  D0-branes. Although it is not a quantum *field* theory, many-particle states are contained in the Hilbert space naturally.

This theory has  $U(N)$  as its symmetry group and a modification of this theory [2] describing type I' theory has  $O(N)$  symmetry<sup>1</sup> group. Apart from unitary and orthogonal groups we know another infinite set of Cartan compact groups: symplectic groups which can be understood as unitary groups over the quaternions:  $USp(2N) = U(N, H)$  in our notation. As far as I know, no matrix model of M-theory with such a symmetry has been described.

While it might be possible to investigate the M-theory from various limits of superstring theories which contain open strings, heterotic strings have no open strings (since left-movers and right-movers are taken from other theories) and therefore no analogies of D0-branes are known for heterotic strings. Therefore we could be afraid of the possibility that just the phenomenologically most interesting limit of the underlying theory – namely heterotic string or equivalently [4] M-theory on  $S^1/Z_2$  – has not a similar non-perturbative microscopic formulation.

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<sup>1</sup>I write  $O(N)$  and not  $SO(N)$  since e.g.  $(-1)$  matrix of  $O(N)$  plays a role of GSO-projection for gauge fermions [11].

In this note I will try to begin to solve the drawbacks of the last two paragraphs. Since I am just an amateur, much more work will be necessary to correct the mistakes and to compute necessary things.

## 2 Modding out a symmetry in superstring theories

Since I will use a similar technique in the case of matrix models, let us first review the corresponding method in the superstring perturbation theory.

The method of “modding out” or “gauging” or “orbifolding” a symmetry can be used to generate various superstring models from other models.

We start with a model which has a subgroup  $\Xi$  of all the operators commuting with hamiltonian. We simply choose a group  $\Xi$  of the symmetries of the system. We will talk about  $\Xi$  also as about the “group of GSO operators”.

Now we “identify” elements of  $\Xi$  with the identity operator. What does it mean? It means at the first place that physical states should be invariant under the elements of  $\Xi$

$$\forall T \in \Xi : \quad T |\psi\rangle = |\psi\rangle. \quad (1)$$

But this is not the whole story. Since the shift of  $\sigma$  coordinate parametrizing a string by  $\pi$  is also the identity, we must add “twisted” sectors where the shift of  $\sigma$  is identified with elements of  $\Xi$ : for each element we have one sector. In the path-integral approach we have even more unified requirements that we must add contributions of all the worldsheets whose action along the noncontractible loop can be identified with elements of  $\Xi$ .

The operators in  $\Xi$  are defined rather formally and their particular definition can differ sector from sector. The rules can be supplied by some computational techniques with diverging sums as in [5] but rules of modular invariance must be obeyed. Modular invariance is a technicality in perturbation superstring theory which has lost its fundamental meaning but now we are entering to the age of a new formulation of the underlying theory where new technicalities may become important.

Although this is quite trivial, let us note different situations which are described by the same idea written above. We may take a usual group of GSO operators counting numbers of some fermionic operators. For instance, if the operator  $T$  changes phases of complex fermions  $f_i$  according to

$$T \cdot f_i = f_i \cdot T \cdot e^{i\phi_i}, \quad (2)$$

where  $\phi_i$  are angles, usually from the set  $0, \pm\pi/2, \pm\pi$  (values  $0, \pm\pi$  are possible even for real fermions), the constraints of the invariance under these symmetries are the usual GSO projections.

What happens in the twisted sectors? The  $\sigma$ -shift by  $\pi$  is an operator we will call  $\Sigma$  and its role is following one:

$$\Sigma \cdot L(0) \cdot \Sigma^{-1} = L(\pi), \quad (3)$$

where  $L$  is an operator being function of  $\sigma$ . If we identify the operator  $\Sigma$  with the  $T \in \Xi$  defined above, we get simply sectors with different boundary phases of fermions:

$$\Sigma \cdot f_i(0) \cdot \Sigma^{-1} = f_i(\pi) = T \cdot f_i(0) \cdot T^{-1} = e^{i\phi_i} f_i(0). \quad (4)$$

There is a large industry of superstring model building (see [6] for instance) where the group  $\Xi$  is taken to be typically  $Z_2^7 \times Z_4$ . Most of these models in the fermionic formulation give three generations of quarks and leptons, often with good quantum numbers, and have many more phenomenological virtues and it is hard to believe that these successes are just accidental although they were obtained in the perturbative theory.

Quite different example of gauging a symmetry, described by the same idea, is compactification on a circle (or more generally, on tori). In this case we take the group  $\Xi$  to be isomorphic to  $Z$  and containing elements

$$T_n = e^{2\pi i n p_i r}, \quad n \in Z \quad (5)$$

shifting a coordinate  $x_i$  by  $nr$ . The condition of the invariance under this group of states simply restricts the total momenta  $p_i$  to be a multiple of  $1/r$ . The twisted sector for the element  $T_n$  contains  $n$ -times winded strings:

$$\Sigma \cdot X_j(0) \cdot \Sigma^{-1} = X_j(\pi) = e^{2\pi i n p_i r} X_j(0) e^{-2\pi i n p_i r} = X_j(0) + 2\pi n r \delta_{ij}. \quad (6)$$

Next good example of this construction is hidden in orbifolds. There are for instance  $Z_3$  symmetries of a torus which can be gauged out.

The symmetry operators can be combined for example with reversion of the  $\sigma$  coordinate and we get orientifolds and so on.

Although the following case is not completely standard, in some sense also open strings can be considered as the twisted sector corresponding to a reversion of  $\sigma$ . If we take  $\Xi$  to be group of the identity and the second element  $T$  reversing  $\sigma$ , it is quite comprehensible that the GSO projection now restricts strings to be unoriented. For the  $T$ -twisted sector the following is true:

$$L(\pi - \sigma) = T \cdot L(\sigma) \cdot T^{-1} = \Sigma \cdot L(\sigma) \cdot \Sigma^{-1} = L(\pi + \sigma) \quad (7)$$

This causes the string to go from the one end to the other and back when we let  $\sigma$  increase. Periodicity becomes  $2\pi$ . By the way, this doubling of the interval for  $\sigma$  to  $2\pi$  is often useful.

If we take type IIB strings and make this operation, one thing must be added: the points  $\sigma = 0$  and  $\sigma = \pi$  are fixed under  $T$  and special things at

these points can be expected. The novelty is the 32 possible colours of the ends. It makes  $SO(32)$  type I strings from type IIB strings. While the necessity of 32 D9-branes is well-established today, I will only offer a similar thing in the present construction, which could generate the  $E_8$  symmetries of the resulting heterotic string matrix model.

### 3 Modding out a symmetry in M(atrix) theory

Let us try to find a similar group  $\Xi$  of operators commuting with the hamiltonian taken<sup>2</sup> from [1]:

$$H = R \cdot \text{tr} \left\{ \frac{\Pi_i \Pi_i}{2} - \frac{1}{4} [X^i, X^j] [X^i, X^j] + \theta^T \gamma_i [\theta, X^i] \right\} \quad (8)$$

We again require the physical states to be invariant under the elements of  $\Xi$ :

$$\forall T \in \Xi : \quad T |\psi\rangle = |\psi\rangle. \quad (9)$$

What will be the counterpart of the twisted sectors? I think that they will be obtained (in the string limit) after the following procedure whose particular example I saw in [3] in connection with compactifications to tori.

We just enlarge  $N$  – the size of the matrices – and we will choose a subgroup  $\Xi'$  of  $U(N)$ , the gauge symmetry group of the matrix model, isomorphic to  $\Xi$ . Then we identify the elements of  $\Xi$  with elements of  $\Xi'$ . This identification is hidden in the restriction of matrices  $X^i, \theta, \Pi$  to satisfy

$$T' \cdot X \cdot T'^{-1} = T \cdot X \cdot T^{-1}, \quad (10)$$

where  $T \in \Xi$  and  $T' \in \Xi'$  are the corresponding elements of groups. On the left hand side there is just action of an element of the group  $U(N)$  in the adjoint representation while on the right hand side there is the physical operation.

An example is the group  $\Xi$  of operators shifting a dimension (that we want to compactify on a circle).

$$T_n = e^{2\pi i n p_i r}, \quad n \in \mathbb{Z}. \quad (11)$$

Then the restriction of  $X$  has the result described in [3]. (I plan to describe other applications in [8].) We can talk about that as about putting the D0-branes to all the identified points.

Now I can also mention that the need of the condition for states to be invariant under the elements of  $\Xi$  is now more clear from the fact that we have identified  $\Xi$  with a subgroup of  $U(N)$  – and physical states certainly must be invariant under all the  $U(N)$ .

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<sup>2</sup>The sign before the squared-commutator term has been changed since I think that the commutator of two hermitian  $X^i$ 's is antihermitian so its square is *negatively* definite.

We may also have a look what happens if we try to identify the identical element of the physical  $\Xi$  with a non-identical element of  $\Xi' \subset U(N)$  (the opposite attempt cannot succeed e.g. for circular compactification, giving conditions like  $x = x + R$ ), say a diagonal matrix. Then we constrain matrices  $X, \theta$  to be block diagonal and we obtain really non-interacting copies of the universe.

I want to mention that these ideas applied for the circular compactification were first realized by Banks and his collaborators [10]. Namely, Tom Banks was the first to say that for achieving compactification we should look at matrices which are gauge equivalent to translations of themselves. After some corrections due to E. Witten this formulation (which we believe to be correct now) was obtained, restricting directly the configuration of matrices instead of states as in the original proposal. Afterwards the team [1] also realized that this prescription should be understood also as an extrapolating of the D0-brane theory in the compactified space from the weakly coupled string theory.

I am grateful to T. Banks for this comment and I apologize for the possible misunderstandings which could result from my text.

## 4 The symmetry reversing spacetime and the membrane

Now we would like to apply this method to M-theory with *one* boundary, where a gauge group  $E_8$  should live as Hořava and Witten showed [4]. So the group  $\Xi$  will be isomorphic to  $Z_2$  containing identity and the operator  $T$  reversing one of the nine transverse coordinates, let us choose  $X^1$ . Physical states should be invariant under the action of  $T$ .

What does  $T$  make with coordinates? It must anticommute with  $X^1$  in order to change its sign while it should commute with  $X^2 \dots X^9$  to let them intact (below also  $T = T^{-1}$ ).

$$T \cdot X^1 \cdot T^{-1} = -X^1, \quad T \cdot X^i \cdot T^{-1} = X^i, \quad i = 2 \dots 9. \quad (12)$$

Spinors should be multiplied by<sup>3</sup> the gamma matrix  $\gamma^1$  of the 16-dimensional real representation of  $spin(9)$ :

$$T \cdot \theta \cdot T^{-1} = \gamma^1 \theta. \quad (13)$$

Let me mention that the  $spin(9)$  gamma matrices are chosen to be real and symmetric. I will use the unified symbol “ $\pm$ ” which is “ $-$ ” for  $X^1$ , “ $+$ ” for  $X^2 \dots X^9$  and “ $\gamma_1$ ” for  $\theta$ ’s. (Gamma matrices have eigenvalues  $\pm 1$ .)

Are the terms in the matrix model hamiltonian [1] invariant under such an operation, changing sign of  $X^1$  (and also  $P^1$ ) and multiplying spinors by  $\gamma_1$ ?

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<sup>3</sup>We can choose  $-\gamma^1$  instead of  $\gamma^1$  but we must choose one of these possibilities. Since  $\gamma^1$  is a chirality operator for  $spin(8)$  rotating  $X^2 \dots X^9$ , we are creating a chiral theory.

While the bosonic terms proportional to  $\Pi_i^2$  and  $[X^i, X^j]^2$  obviously are, the fermionic term requires a careful counting of signs:

$$\text{tr } \theta^T \gamma_i [\theta, X^i] \mapsto \text{tr } \theta^T \gamma_1 \gamma_i [\gamma_1 \theta, \pm X^i] \quad (14)$$

For  $i = 1$  the three  $\gamma_1$  matrices can be reduced to one but  $X^1$  changes the sign – so the total contribution changes the sign.

For  $i > 1$  due to the anticommutation relations  $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$  the two  $\gamma_1$ 's can be transferred to each other ( $(\gamma_1)^2 = 1$ ) but it changes the sign. Since  $X^i$  is invariant, also in this case the total contribution changes the sign.

So whole the last term changes the sign under our operation. So our operation is not complete symmetry of the hamiltonian. We should multiply it by some next operation under which the first two terms are even and the last term is odd.

Such an operation exists. Let me say immediately that this operation is transposition of all the matrices – or equivalently (because of their hermiticity) – their complex conjugation. (For operators I mean that each element of the matrices is hermite-conjugate.)

The bosonic terms are quite obviously invariant under the transposition of matrices. The fact that transposition changes the sign of the last term requires a careful counting of signs. Let us write the trace using spinor indices  $\alpha, \beta$ , Lorentz-vector index  $i$  and  $U(N)$  indices  $k, l, m$ :

$$\theta_{kl}^\alpha \gamma_i^{\alpha\beta} (\theta_{lm}^\beta X_{mk}^i - X_{lm}^i \theta_{mk}^\beta) \quad (15)$$

If we transpose the matrices – which corresponds to the transposition of their indices e.g.  $k, l$ , we get

$$\theta_{lk}^\alpha \gamma_i^{\alpha\beta} (\theta_{ml}^\beta X_{km}^i - X_{ml}^i \theta_{km}^\beta) = \theta_{lk}^\alpha \gamma_i^{\alpha\beta} (-\theta_{km}^\beta X_{ml}^i + X_{km}^i \theta_{ml}^\beta) \quad (16)$$

the opposite sign for the result compared to starting formula.

## Where do the membranes end?

Let us forget for a while the  $\Xi'$  being the subgroup of  $U(N)$  and study the formula (8.2) in [1] combined with our requirement for states to be invariant under  $T$  – the symmetry combining transposition of matrices and reversion of  $X^1$ . The formula (8.2) of [1] reads (we use it for representing matrix  $X^2$  as our example):

$$X^2 = \sum_{m,n=1-[N/2]}^{N-[N/2]} Z_{mn} U^m V^n \exp(-\pi i mn/N). \quad (17)$$

I added the phase to symmetrize the order in which the noncommuting operators  $U, V$  are written. It has the virtue of better properties for various conjugations (see below) and its drawback is changing the sign after  $m \rightarrow m + N$  for odd

$n$  and vice versa. Nevertheless, for a low energy membranes the contributions with  $|mn| < N$  are the most important and here the phase factor differs from 1 only a little.

Now we require the states to be invariant under  $T$ .  $T$  has no effect to  $X^2$  so it reduces effectively to the transposition. Let us write a particular form of the “clock” and the “shift” operators:

$$U = \begin{pmatrix} 1 & & & \\ & e^{2\pi i/N} & & \\ & & e^{4\pi i/N} & \\ & & & \ddots \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \dots & 1 \\ 1 & 0 & \\ 1 & 0 & \\ 1 & \dots \end{pmatrix} \quad (18)$$

Clearly,  $U$  is symmetric and the transposition of  $V$  is  $V^{-1}$ . That means that the transposition inverts one of the matrices ( $V$ ). Alternatively, if we use the complex conjugation,  $V$  is real and the complex conjugate of  $U$  is  $U^{-1}$ .

In both cases, the operation inverts one of the two matrices. To be concrete, let us talk about the transposition. Using the facts just stated it is easy to show that

$$(U^m V^n e^{-(\pi imn/N)})^T = e^{-(\pi imn/N)} V^{-n} U^m = U^m V^{-n} e^{\pi imn/N} \quad (19)$$

after the transposition in the contribution to the  $X^2$  proportional to  $Z_{mn}$  the remaining factor will be replaced by the factor which was associated to  $Z_{m,-n}$  before the transposition. Such a changing of Fourier mode  $n$  to  $-n$  is in the continuous basis equivalent to reversion of one coordinate on the fuzzy torus-like membrane. Thus the condition for invariance of the states i.e. for the symmetry of  $X^2$  under the transposition (let us suppose an eigenstate of matrix elements of  $X^2$  and understand  $X^2$  as a classical matrix) tells us something like

$$X^2(p, -q) = X^2(p, q). \quad (20)$$

For the  $X^1$  coordinate changing the sign included in  $T$  will change the formula to

$$X^1(p, -q) = -X^1(p, q), \quad (21)$$

which means that the membrane ends with its boundary  $q = 0$  on  $X^1 = 0$ :  $X^1(p, 0) = 0$ . The same is true for  $q = \Delta/2$  where  $\Delta$  is the period of  $q$  in the fuzzy torus. Here there is the second boundary and the torus is restricted to a cylinder.

Now we could be afraid of the fact that the restricting operators  $X, \theta$  will change this result. I do not think so because the role of these restrictions of operators can be understood as the freedom to produce the twisted sectors and we can always think about a “quite an isolated world” located in the part of the matrices where elements of  $\Xi'$  look the same (in one of the blocks).

In fact, the argument of this section is more reliable in the orthogonal case (using  $\sigma^3$  into  $T'$  below) than in the symplectic one. Nevertheless, we have

showed that using real symmetric matrices (instead of complex hermitean ones) restricts the fuzzy torus to be the fuzzy cylinder. Thus I disagree with various recent claims expressing the absence of open membranes in the non-commuting torus construction and the need to add some boundary terms.

### Möbius and Klein bottle membranes

I studied the question a little. The transposition or the complex conjugation produces the cylinder, mapping

$$U, V \mapsto U, V^{-1} \quad \text{or} \quad U^{-1}, V. \quad (22)$$

We could obtain also the Möbius strip in a similar way. The only thing we must practice is the corresponding mapping

$$U, V \mapsto V, U. \quad (23)$$

The fact that it produces a Möbius strip is clear from the picture I cannot put here. But the usual representation of the strip – the square with a pair of distinct opposite boundaries and the second pair antiidentified – is obtained here as a square of  $1/2$  area compared to the torus and is by 45 degrees rotated. If you draw this smaller square into the original (torus) square which is divided by the symmetry around the axis  $x = y$  and move it by  $\Delta/2$  above, you will understand why it has the Möbius topology.

We can obtain such a mapping by a minor modification of the conjugation  $W \mapsto \bar{W}$  for  $W = U, V$ , namely by adding a discrete Fourier transformation ( $\omega = \exp(2\pi i/N)$ ):

$$W \mapsto F \cdot \bar{W} \cdot F^{-1}, \quad \text{where} \quad F = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & \omega & \omega^2 & \omega^3 & \dots \\ 1 & \omega^2 & \omega^4 & \omega^6 & \dots \\ 1 & \omega^3 & \omega^6 & \omega^9 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (24)$$

But I do not know where this matrix could appear. (Note that  $F^2$  is an antidiagonal matrix and  $F^4 = 1$ .)

Finally, if we would restrict the matrix representing the membrane to be invariant under the mapping of the type

$$U, V \mapsto U^{-1}, -V, \quad (25)$$

we could get a membrane with the Klein bottle topology. It differs from the cylinder only by the minus sign in  $-V$ . This change denotes a shift by  $\Delta/2$  so the membrane should be invariant under the combined operation of reversing one coordinate and shifting the second by  $\Delta/2$ . Clearly, one pair of opposite sides is still identified and the second pair is anti-identified, giving a standard representation of the Klein bottle.

## 5 Gauging this symmetry and the appearance of quaternions

In the previous section we were discussing a pleasant result of the required invariance of states. Now we would like to study the restriction of operators. We must choose an element  $T'$  of  $U(2N)$  (since now we take the size of matrices to be even) which will represent  $T \in \Xi$ . Since also a transposition plays the game, all the choices will not be completely equivalent. Let us suppose that  $T'$  is a tensor product of unit matrix and some  $2 \times 2$  matrix whose square is also the unit matrix. I did not want to use a trivial one (again a unit matrix) – so the most natural candidates are Pauli matrices.  $\sigma^1$  and  $\sigma^3$  give some results (see the “orthogonal case”) but the most interesting seemed to me to use the imaginary Pauli matrix –  $\sigma^2$ . I could not say why this choice was better than others, but I felt it from the resulting symmetry. (Now I think that choosing  $\sigma^1$  or equivalently  $\sigma^3$  gives the more interesting and realistic theory which I will shortly discuss later.) Let  $T'$  be the block diagonal matrix consisting of  $\sigma^2$ 's on the block diagonal.

$$T' = \text{diag}(\sigma^2, \sigma^2, \sigma^2, \dots) \quad (26)$$

Let  $Y$  denote  $X^{2\dots 9}$  or  $X^1$  or  $\theta$ 's and  $\pm$  is minus for  $X^1$ , plus for the remaining  $X$ 's and  $\gamma_1$  for  $\theta$ 's. We require as in [1] all the  $Y$ 's being hermitean complex matrices. Let us write the requirement for  $Y$ 's:

$$T' \cdot Y \cdot T'^{-1} = \pm \bar{Y}, \quad (27)$$

where  $\bar{Y}$  means complex conjugation (i.e. hermitean conjugation of matrix's elements). Those  $\sigma^2$ 's in  $T'$  act on each  $2 \times 2$  block of  $Y$  giving a restriction for it:

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} = \pm \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} \quad (28)$$

so the conditions are  $D = \pm \bar{A}$ ,  $-C = \pm \bar{B}$ . Let us suppose for a while that  $\pm = +$ . Then the conditions constrain the  $2 \times 2$  blocks of  $Y$ 's to be of form ( $a, b, c, d$  are real)

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \quad (29)$$

Similarly, for  $\pm = -$  we require the  $2 \times 2$  blocks to be  $i$  times the matrix of the type above. For spinors  $\pm = \gamma_1$  but this is nothing new: half of components have  $\pm = \gamma_1 = +1$  and half of them have  $-1$ . The  $2 \times 2$  matrices of the form above have exactly the same multiplication rules as quaternions  $a + bi + cj + dk$ . And also the hermitean conjugation of such a matrix gives the conjugate quaternion  $a - bi - cj - dk$ . Thus we can replace these blocks by quaternions.

The situation  $\pm = -$  differs in one basic aspect only. We can again consider this as a representation of quaternions (where  $i$  or  $-i$  – I cannot decide now –

must be added to each product) but their hermitean conjugation gives *minus* conjugate quaternion  $-a + bi + cj + dk$  in the same representation.

This mean that  $X^2 \dots X^9$  (and half spinors  $\gamma_1 = 1$ ) can be regarded as hermitean quaternionic matrices (hermitean conjugation for quaternionic matrices is a combination of transposition and quaternionic conjugation) while  $X^1$  (and the rest of spinor components) as antihermitean quaternionic matrices.

In a process of a typical orbifolding, the  $U(2N)$  symmetry would be reduced to  $U(N) \times U(N)$  symmetry. Now, because of the complex conjugation, the symmetry would be restricted to  $U(N)$  in a generic case. Therefore this generic case should be inconsistent since (intuitively) the consistent theories should have about the same dimension of the gauge group as one of  $X_i$ 's, say  $X_1$ . (Note for instance that all the groups  $U(N) \times U(N)$ ,  $O(2N)$  and  $USp(2N)$  have dimension about  $2N^2$ .) Thus only extremal choices have a chance to be consistent. Giving  $\sigma^2$  to  $T'$  leads to the quaternionic theory while using  $\sigma^1$  or equivalently  $\sigma^3$  should create a theory with orthogonal symmetry. In the quaternionic case, the  $T'YT'^{-1} = \bar{Y}$  corresponds to  $jYj^{-1}$  and the invariance under this  $Z_2$  combines with the generic  $U(N)$  to the requirement of the invariance under whole  $U(N, H)$ .

Now we could be surprised by the antihermitean form of  $X^1$ . The remaining coordinates are correct and have real numbers on the diagonal. But  $X^1$  has “purely quaternionic” numbers  $bi + cj + dk$  on its diagonal. It seems as the  $X^1$  coordinate exists three times.

But fortunately, this is not the case. The reason is that directions in the three-dimensional space of numbers  $bi + cj + dk$  are all equivalent since they can be transformed to each other (due to the noncommutativity of quaternions) by the transformations of  $USp(2N) = U(N, H)$  ( $H$  denotes the set of quaternions and the matrices  $Y$ 's are taken to be  $N \times N$  quaternionic). In fact, these directions are equivalent even to their opposite. But even this should not be too big surprise since the sign of  $X^1$  coordinate is unphysical.

Even in the potential case where we would use more than one ( $X^1$ ) antihermitean quaternionic matrix, no problem would arise because these coordinates would have on the corresponding sites of diagonal typically “pure imaginary quaternions” from the same direction - i.e. real-number-proportional to each other. In the opposite case the energy containing commutators would increase rapidly again due to the noncommutativity of quaternions:

$$[i, j] = 2k, \quad [i, j]^2 = -4 \quad (30)$$

So we just say that a formulation of the M-theory on a space with boundary requires quaternionic matrices  $X, \Pi, \theta$  where  $X^1$  and half of  $\theta$ 's are antihermitean while the other are hermitean and the hamiltonian looks essentially as in [1]. Perhaps, new degrees of freedom – perhaps again in the fundamental representation (natural from the point of view that it is associated to  $N$  elements of boundary of the membrane which has  $N^2$  elements because it is associated with

matrix) of the  $U(N, H)$  symmetry group – should be added as in [2] together with terms in the hamiltonian

$$\sum_{r=1}^{8 \text{ or } 16?} \lambda_r^{k\dagger} X_{kl}^1 \lambda_r^l \quad (31)$$

Their existence could be explained by similar arguments concerning the transposition of matrices as the origin of 32 colours at the ends of type I string was explained as a side-effect of reversion of  $\sigma$ . Also, a prematrix theory might be found where elements of  $X \dots$  operators would be represented as states, restrictions of these operators as restrictions of these states and the new 16  $O(N)$  vectors would arise from the fixed points of a  $Z_2$  operation in a similar way as in description of (0,1) heterotic strings by (2,1) strings [7]. (In fact, I was trying to obtain the fields necessary for the gauge symmetry from the original spinors so that they would loose their Lorentz quantum numbers but this is perhaps a lousy idea.) It is quite interesting because some papers indicate that it should be possible to get all the compactifications from the original M-theory without adding degrees of freedom. Maybe that this result is limited to theories which originate from *untwisted* algebra [7] of (2,1) heterotic strings.

### A short description of the $U(1, H)$ system

In [1], the most simple case  $N = 1$  with the symmetry  $U(1)$  gave a free theory with 256 states having a momentum. In the quaternionic case, the simplest case has symmetry  $USp(2) = U(1, H)$  which is isomorphic to  $spin(3)$ .  $X^2 \dots X^9$  and  $\Pi^2 \dots \Pi^9$  as well as  $\theta$ 's with  $\gamma_1 = 1$  are hermitean  $1 \times 1$  quaternionic operator-matrices - so they are hermitean scalars. But the remaining components of  $\theta$  and  $X^1$  as well as  $\Pi_1$  are *antihermitean*, so they have the form  $bi + cj + dk$  and transform as vectors under the  $spin(3)$  group.

Now we would like to see if the physics in the bulk of [1] is reproduced in this model. Let us begin with an eigenstate of  $Y^1$  components. We can make a  $spin(3)$  transformation to achieve

$$Y_x^1 = Y_y^1 = 0. \quad (32)$$

The real physical states invariant under the  $spin(3)$  are then obtained by the integration over all the group. The hamiltonian looks like

$$H = \sum_{i=2}^9 \frac{\Pi_i \cdot \Pi_i}{2} + \sum_{j=x,y,z} \frac{\Pi_1^j \cdot \Pi_1^j}{2} + \sum_{r=1}^8 i \varepsilon_{ijk} \cdot X_i^1 \theta_j^r \theta_k^r. \quad (33)$$

The momenta  $2 \dots 9$  contribute to the energy in the same way as in [1] and the same is true also for  $\Pi_1$  when  $|Y^1|$  is large. The components of  $\theta$  with  $\gamma_1 = +1$  are not contained in hamiltonian – in the same fashion as in [1] where they ensure (together with the  $\gamma_1 = -1$  components) the 256 degeneracy of states.

Where are the  $\gamma_1 = -1$  components in our construction? Because of the  $\varepsilon_{ijk}$ , for  $Y^1$  having the  $z$ -direction the  $z$ -components of spinors with  $\gamma^1 = -1$  decouple from the hamiltonian and just these states play the role of the scalar  $\theta$  components with  $\gamma^1 = -1$  in [1].

But the  $x$  and  $y$  components of  $\theta_{\gamma_1=-1}$  are interacting. It is natural to combine them to combinations  $\theta_x \pm i\theta_y$ . The ground level is annihilated by all  $\theta_x + i\theta_y$  (or minus?) and is a Lorentz scalar since the sum of weights in any representation equals zero. But such a ground level is not a  $spin(3)$  scalar since it has  $j_z = -4$ . (The opposite ground level obtained by application of all the eight creation operators  $\theta_x - i\theta_y$  must have  $j^z$  greater by 8 and at the same moment, inverse to the  $j_z$  of the true ground level.)

So there is an anomaly. Its result is by the way also an energy proportional to  $Y^1$ . I think that the most natural way to cancel this anomaly is to add fermions in the fundamental representation of  $spin(3)$  i.e. spinors together with a hamiltonian term like

$$\sum_{s=1}^8 \sum_{p,p'=1,2} \lambda_s^{\dagger p} \sigma_{pp'}^i X_i^1 \lambda_s^{p'} \quad (34)$$

We need 8 such spinors because each of these have two components but they have only  $j_z = \pm 1/2$ . I have thought for a time that these fermions can be the source of the  $E_8$  symmetry but I found that they can generate symplectic symmetries much more easily than  $SO(16)\dots$

The spin-statistics theorem is obeyed because of the  $j = 0$  condition: the only variable which could break it ( $\lambda$ ) has  $j = 1/2$  so its creation operators must be always paired. Nevertheless, I do not know if all these ideas can lead to a really consistent theory... Only now I realized that maybe it's more natural to add bosons (and not fermions  $\lambda$ ) but I leave this question to a future work.

## The orthogonal choice

I just repeat the discussion from the last section for  $\sigma^1$ :

Those  $\sigma^1$ 's in  $T'$  act on each  $2 \times 2$  block of  $Y$  giving a restriction for it:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix} = \pm \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} \quad (35)$$

so the conditions are  $D = \pm \bar{A}$ ,  $C = \pm \bar{B}$ . Let us suppose for a while that  $\pm = +$ . Then the conditions constrain the  $2 \times 2$  blocks of  $Y$ 's to be of form ( $a, b, c, d$  are real)

$$\begin{pmatrix} a + bi & c + di \\ c - di & a - bi \end{pmatrix} \quad (36)$$

Similarly, for  $\pm = -$  we require the  $2 \times 2$  blocks to be  $i$  times the matrix of the type above. For spinors  $\pm = \gamma_1$  but this is nothing new: half of components have  $\pm = \gamma_1 = +1$  and half of them have  $-1$ .

The matrix above can be written as ( $a, b, c, d$  are real)  $a + ib\sigma^3 + c\sigma^1 - d\sigma^2$  and is equivalent to  $a + ib\sigma^2 + c\sigma^x + d\sigma^3$  which is real. Note also that the hermitean conjugation makes the same operation in both cases:  $b \mapsto -b$ .

A similar result we would get for  $\sigma^3$  and in fact also  $1_{(2 \times 2)}$  gives real blocks. In the latter case we see clearly that  $X^1, \Pi^1$  and  $\gamma_1 = -1$  spinors are antisymmetric purely imaginary<sup>4</sup> matrices while the others are symmetric real as in [2]. So these prescriptions require  $X$ 's and  $\theta$ 's to be real (elements to be hermitean) and the symmetry  $U(2N)$  is restricted to  $O(2N)$ . Now I think that this orthogonal case may lead to a standard  $E_8$  symmetry on the boundary after adding a vector of  $SO(16)$  tensored with a vector of  $O(2N)$ . In fact I see no differences between the recent model and the model of [2]. So I can say the most visible result of [2] that the states 120 are represented in the  $O(2N)$  system while the remaining states 128 of  $E_8$  are included in the  $O(2N+1)$  (which is possible if we choose unit matrix instead of Pauli ones). For instance, for the  $O(1)$  system, all the antisymmetric matrices ( $X^1, \Pi^1$  and spinor components with  $\gamma_1 = -1$ ) must equal to zero – so the states are living at  $X^1 = 0$  – and only operators  $X^i, \Pi_i, i = 2 \dots 9$  and spinor components of  $\gamma_1 = +1$  plays the role, giving the standard  $16_{\text{from spinors}} \times 256_{\text{from } \lambda's} / 2_{\text{from the projection}}$  degeneracy of the states. I plan a paper [11] on bosonic  $E_8$  and some corrections of [2].

## 6 Conclusions

In this note I was trying to obtain a matrix model formulation for M-theory on a space with boundary. The orbifolding of  $Z_2$  symmetry seems to give two apparently consistent possibilities. The orthogonal one restricts matrices to be equivalent to real matrices and gives  $O(2N)$  symmetry (this choice was forgotten in the first version of the paper) or even  $O(N)$  symmetry not only for even  $N$ 's. Now I believe that just the missed orthogonal case describes the M-theory with  $E_8$  on one boundary [2] originally found from type I' D0-branes in the same sense as the full M-theory on  $M^{11}$  was obtained from type IIA D0-branes [1].

The second – symplectic version may give another consistent theory or even (if the added degrees of freedom are bosons) the same theory with the bosonic representation of  $E_8$ . In this one quaternionic matrices appeared quite naturally. This theory may give also a different consistent system in 11 dimensions – with the boundary lived e.g. by  $USp(16) \times SO(8)^4$  multiplet. (Symplectic symmetries arise in the quaternionic case much better than the orthogonal ones.) This theory would not be described by any limit of any string theory known. The real and quaternionic cases have something common: complex representations represent a group as a subgroup of  $U(N)$  and are not equivalent to their complex conjugates while the real (subgroup of  $O(N)$ ) and pseudoreal=quaternionic (subgroup of  $USp(2N)$ ) are.

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<sup>4</sup>Multiplying by  $i$  gives real antisymmetric matrices. We keep the convention of hermiticity for all the matrices.

It should be verified if the new physical system satisfies a correct supersymmetry algebra [2,11,future]. Also the idea should be generalized to compactification to  $S^1/Z^2$  [8,future]; now we were only briefly discussing orbifolding to  $R/Z^2$  where only one e.g.  $E_8$  appears. Physical states in the “bulk” of the spacetime should be the same as in [1] and new states (super-Yang-Mills  $E_8$  multiplet) associated to the boundary should be found.

If the quaternionic theory would appear consistent, we would have theories with all the possible gauge groups from the infinite sets of simple groups. Even  $U(m, n)$  symmetry group has been used to study brane-antibranes interaction and the analytical continuation from  $U(m + n)$  was showed to correspond to crossing symmetry [9].

I appologize for my poor English and I wish you a M(erry) Christmas and a Happy New Year.

## References

1. T.Banks, W.Fischler, S.H.Shenker, L.Susskind: *M Theory As A Matrix Model: A Conjecture*, hep-th/9610043
2. S.Kachru, E.Silverstein: *On Gauge Bosons in the Matrix Model Approach to M Theory*, hep-th/9612162, the model was first written down in U.Danielsson, G.Ferretti: *The Heterotic Life of the D-particle*, hep-th/9610082
3. W.Taylor: *D-brane field theory on compact spaces*, hep-th/9611042
4. P.Hořava, E.Witten: *Heterotic and Type I String Dynamics from Eleven Dimensions*, hep-th/9510209
5. L.Motl: *Two-parametric zeta function regularization in superstring theory*, hep-th/9510105
6. A.Faraggi: *Realistic Superstring Models*, hep-ph/9405357
7. D.Kutasov, E.Martinec: *New Principles for String/Membrane Unification*, hep-th/9602049
8. L.Motl: *Proposals on nonperturbative superstring interactions*, hep-th/9701025
9. V.Periwal: *Antibranes and crossing symmetry*, hep-th/9612215
10. T.Banks: speech at Aspen Workshop, summer 1996  
and Princeton, September 1996, unpublished
11. L.Motl: *Bosonic representation of gauge symmetry in M(atrix) theory*, hep-th eprint in preparation